

A metric characterisation of repulsive tilings

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Abstract

A tiling of \mathbb{R}^d is *repulsive* if no r -patch can repeat arbitrarily close to itself, relative to r . This is a characteristic property of aperiodic order, for a non repulsive tiling has arbitrarily large local periodic patterns.

We consider an aperiodic, repetitive tiling T of \mathbb{R}^d , with finite local complexity. From a spectral triple built on the discrete hull Ξ of T , and its Connes distance, we derive two metrics d_{sup} and d_{inf} on Ξ . We show that T is repulsive if and only if d_{sup} and d_{inf} are Lipschitz equivalent. This generalises previous works for subshifts by J. Kellendonk, D. Lenz, and the author.

1 Introduction

In two recent articles in collaboration with J. Kellendonk and D. Lenz [11, 12], we used constructions of non commutative geometry [1] to derive a new characterisation of aperiodically ordered $1d$ -subshifts. We showed that a minimal and aperiodic subshift X has bounded powers if and only if two metrics derived from the Connes distance of a spectral triple over X are Lipschitz equivalent. An essential ingredient to obtain this result is the notion of *privileged words* [12]. In this paper, we generalise this formalism and this results to tilings of \mathbb{R}^d . The essential ingredient here is the notion of *privileged patches* of a tiling.

A $1d$ -subshift has bounded powers if its language does not contain arbitrarily large powers, *i.e.* there is an integer p such that n -fold repetitions $w^n = w \cdots w$ of a word w cannot occur for $n > p$. Linearly recurrent subshifts, which are usually considered highly ordered, share this property [13, 4, 5]. Loosely speaking, bounded powers means that no factor can repeat too close, or overlap too much, along a sequence in the subshift. Bounded powers is equivalent to the property that a complete first return u' of a word u must be longer than a uniform constant times the length of u : $|u'| > C|u|$. The corresponding notion for tilings is *repulsiveness*: no patch can repeat arbitrarily close to itself relative to its size, see equation (1). A non repulsive tiling has arbitrarily large local periodic patterns – the analogue of arbitrarily large powers. As for subshifts, linearly repetitive tilings are repulsive [15, 14].

The property of bounded (or unbounded) powers in a subshift is measured by privileged words. Privileged words are iterated complete first returns to letters of the alphabet. Privileged words were introduced in [12], and have recently encountered a lot of interest in the combinatorics of words [17, 10, 18, 9]. For rich subshifts [6] privileged words coincide exactly with palindromes (see [12] Section 2.2 for further details).

We generalise this notion to tilings. We define privileged patches: a notion of iterated complete first returns to the prototiles, see Section 3. For $1d$ subshifts, a privileged patch is a generalisation of a privileged word obtained with bilateral versions of complete first returns. Because of the geometry in \mathbb{R}^d , the combinatorics of patches is much more involved than that of words. We need a few technical lemmas to deal with this. But the crucial point is the generalisation of privileged words to the tilings setting. Once the right definition of privileged patch is at hand, our formalism for subshifts essentially goes through for tilings of \mathbb{R}^d . The spectral triple we used in [12] for subshift is build from the tree of privileged words of the subshift. The spectral triple we use here is the same one built on the tree of privileged patches of the tiling. This allows us to characterise repulsive tilings by Lipschitz equivalence of two metrics derived from the Connes distance of the spectral triple, in complete analogy with the case of subshifts treated in [12].

Our initial motivation in studying properties of aperiodically ordered subshifts and tilings, came from non commutative geometry (NCG) [1]. Namely we were interested in the construction of non commutative Riemannian structures, *i.e.* spectral triples, over totally disconnected spaces defined by tilings and subshifts. As it turns out, and as in [12], the criterium for aperiodic order we derive here can be explained in a rather combinatorial way, without introducing the framework of NCG and giving the details of the construction of the spectral triple. So we follow this line in the paper: we give the criterium *ad hoc* to state and prove our result. And in the last section we describe briefly the underlying spectral triple.

The paper is organised as follows. In Section 2 we remind the reader of the basic definitions for tilings of \mathbb{R}^d , and the classical results we need. We introduce privileged patches in Section 3, and state some combinatorial properties, including technical lemmas which allows us to adapt our formalism for subshifts to tilings of \mathbb{R}^d . In Section 4 we explain the construction of the tree of privileged patches, from which we define the two Connes metrics. In Section 5 we state and prove our main result, namely that a tiling is repulsive if and only if the Connes metrics are Lipschitz equivalent. The construction of the spectral triple, from which the Connes metrics are derived, is given briefly in Section 6.

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2 Basic definitions

A *tile* of \mathbb{R}^d is a subset $t \subset \mathbb{R}^d$ which is homeomorphic to a closed ball. A *tiling* of \mathbb{R}^d , is a countable family of tiles, $T = \{t_i\}_{i \in \mathbb{N}}$, which have pairwise disjoint interiors and whose

union covers \mathbb{R}^d . Given a tiling T , we specify a *marker*¹ in each of its tile t : a point $x(t) \in \mathbb{R}^d$ in its interior.

A *translate* of a family $F = \{t_j\}_{j \in J}$ of tiles of T , is a family $F + a = \{t_j + a\}_{j \in J}$, for some $a \in \mathbb{R}^d$. Let x be the marker of a tile of T , and $r > 0$. We call an *r-patch*, or a *patch* of radius r , the finite family of tiles of $T - x$ all of whose markers lie inside the open ball $B(0, r)$. In addition, r is maximal with respect to the family of tiles defining the patch. As a consequence, the only 0-patch is the empty patch. The patches made of a single tile (containing the marker of a single tile), are called *prototiles*.

Consider an r -patch p of T . Given a family $F = \{t_j\}_{j \in J}$ of tiles of T , we say that p *occurs in* F , if there is a translate of p which is a subset of F : $p + a \subset F$ for some $a \in \mathbb{R}^d$. The translate $p + a$ is called an *occurrence* of p in F . Given a subset U of \mathbb{R}^d , we say that p *occurs in* U , if there is an occurrence of p in T , the union of all of whose tiles is a subset of U . We mean that a patch p is marked at the origin: $x(p) = 0$. And that an occurrence of p in T , in a family of tiles F , or in a subset of \mathbb{R}^d , is some translated copy $p + a$ marked at a : $x(p + a) = a$.

We will consider tilings satisfying the following three properties.

Definition 2.1. A tiling T of \mathbb{R}^d is called

- (i) *aperiodic* if $T + a = T$ implies $a = 0$;
- (ii) *repetitive* if for any $r > 0$, and any r -patch p of T , there exists $R > 0$ such that p occurs in any ball of radius R ;
- (iii) *FLC*, or has Finite Local Complexity, if for any $r > 0$ there are finitely many r -patches.

Let T be a repetitive and FLC tiling of \mathbb{R}^d , and p an r -patch of T . The *Delone set of occurrences of p in T* is the set L_p of markers of all occurrences of p in T . This is a Delone set as the distance between nearest points of L_p is uniformly bounded. We let $r_{\text{pack}}(L_p)$ (resp. $r_{\text{cov}}(L_p)$) be one half of that uniform minimal distance (resp. maximal distance). It is called the *packing radius* of L_p (resp. *covering radius*): any ball of radius $r_{\text{pack}}(L_p)$ (resp. $r_{\text{cov}}(L_p)$) contains at most (resp. at least) one point of L_p . A tiling T is said to be *repulsive* if

$$\ell = \inf \left\{ \frac{r_{\text{pack}}(L_p)}{r} : p \text{ an } r\text{-patch of } T \right\} > 0. \quad (1)$$

Informally, $\ell > 0$ means that patches cannot overlap too much. On the contrary, in a non-repulsive tiling, there are arbitrarily large r -patches with arbitrarily close occurrences relative to r . Such occurrences overlap over an arbitrarily large proportion of their tiles. This implies that a non-repulsive tiling has arbitrarily large local periodic patterns, see Figure 2.

We now fix an aperiodic, repetitive, and FLC tiling T of \mathbb{R}^d and assume that there is a tile whose marker lies at the origin. We endow the family of all of its translates, $T + \mathbb{R}^d$,

¹sometimes also called a *puncture*, so that one talks about *punctured tilings*.

with the following topology. A base of open sets is given by the acceptance domains of patches: for p an r -patch of T

$$[p] = \{T' \in T + \mathbb{R}^d : p \text{ occurs at the origin in } T'\}$$

If q is a patch contained in p , which we write $q \subseteq p$, then $[p] \subset [q]$. Hence two tilings are close for this topology, if they agree on a large patch around the origin. The *discrete hull* of a T is the closure of its translates in this topology:

$$\Xi = \overline{\{T + a : a \in \mathbb{R}^d, T + a \text{ has a marker at the origin}\}}.$$

As a consequence of the hypothesis in Definition 2.1 the following classical results hold:

- Ξ is a Cantor set (compact, totally disconnected, with no isolated point);
- the family of acceptance domains $[p]$ is a countable base of clopen sets² for Ξ ;
- any $T' \in \Xi$ satisfies the hypothesis of Definition 2.1, and the closure of $T' + \mathbb{R}^d$ is Ξ .

The discrete hull is metrizable. Any function $\delta : [0, +\infty) \rightarrow (0, 1]$, which decreases and has limit 0 at $+\infty$, defines a ultra-metric on Ξ as follows:

$$d_\delta(T_1, T_2) = \inf\{\delta(r) : \text{there is an } r\text{-patch } p \text{ occurring in both } T_1 \text{ and } T_2 \text{ at the origin}\}. \quad (2)$$

3 Privileged patches

Given an r -patch p , we say that an r' -patch p' is *derived* from p if

- (i) p is contained in p' ;
- (ii) p occurs at least twice in p' ;
- (iii) for any $\tilde{r} < r'$, and any \tilde{r} -patch q contained in p' , p occurs at most once in q .

See Figure 1 for an illustration. Condition (ii) means that p' contains p as a subpatch (hence with marker at the origin), plus another translate $p+a$ for some $a \neq 0$. Conditions (ii) and (iii) mean that p' is a minimal extension of p containing two occurrences of p .

We define *privileged patches* inductively, as follows:

- (0) the empty patch is the only privileged patch of order 0;
- (1) the prototiles of T , are the privileged patches of order 1;
- (n) for $n > 1$ a privileged patch of order n is an n -th iterated derived patch from the empty patch.

²closed and open sets

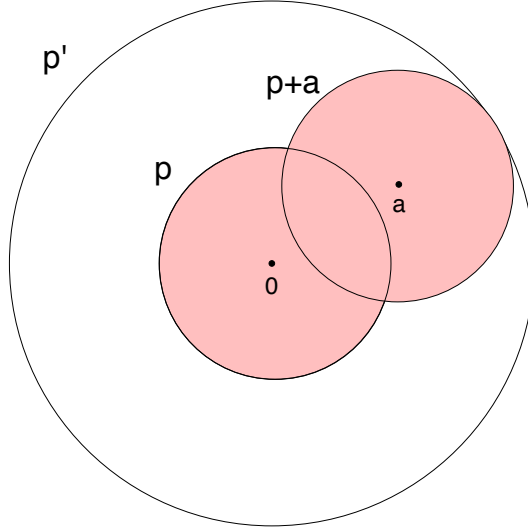


Figure 1: A patch p' derived from p .

For 1d-subshifts, *i.e.* symbolic one-dimensional tilings, this is a two-sided version of *privileged words* introduced in [12].

Let us state some elementary properties of derived patches. The first two Lemmas are needed to build the tree of privileged patches in the next Section.

Lemma 3.1. *Let q be a patch derived from some patch, then*

- (i) *there exists a unique patch p such that $p' = q$;*
- (ii) *if q is privileged, then there exists a unique privileged patch p such that $p' = q$;*
- (iii) *if q is privileged, and p is a privileged patch contained in q , then there exists $i \geq 0$ such that q is an i -th iterated derived patch from p , which we write $p^{(i)} = q$.*

Proof. (i) Assume that $q = p'_1 = p'_2$, for two distinct patches p_i of radius r_i , $i = 1, 2$. We may assume $r_2 < r_1$, but then $p_2 \subsetneq p_1$, and this implies $p'_2 \subsetneq p'_1$ a contradiction.

(ii) If q is privileged, by definition there exists a privileged patch p such that $p' = q$, and by (i) p is unique.

(iii) We prove this by induction on the radius of q . Let $(r_n)_{\mathbb{N}}$ be the non-decreasing sequence of radii of privileged patches of T (which exists by FLC). The property is obvious for privileged patches of radius r_1 : q is a prototile with smallest radius and is derived for the empty patch. Assume the property holds for all privileged patches of radii less than or equal to r_n , for some $n > 1$. Consider a privileged patch q of radius r_{n+1} , and a privileged patch $p \subseteq q$. The case $p = q = p^{(0)}$ is trivial, so assume $p \subsetneq q$. By (ii) there exists a unique privileged patch $\tilde{p} \subsetneq q$ with $\tilde{p}^{(1)} = q$. Case $p \subseteq \tilde{p}$: by induction $\tilde{p} = p^{(j)}$ for some $j \geq 0$, and so $q = p^{(j+1)}$. Case $\tilde{p} \subsetneq p$: by induction there is a $j > 0$ such that $p = \tilde{p}^{(j)} = q^{(j-1)}$, which implies $q \subseteq p$ a contradiction. \square

Lemma 3.2. (i) Let p be an r -patch, and p' an r' -patch derived from p , then

$$2r_{\text{pack}}(L_p) + r \leq r' \leq 2r_{\text{cov}}(L_p) + r.$$

(ii) Let $(p_n)_{n \geq 1}$ be a sequence of r_n -patches, such that p_{n+1} is derived from p_n for all n . Then $r_{n+1} \geq 2nr_{\min}$, where r_{\min} is the radius of the smallest prototile. If in addition T is repulsive, then $r_{n+1} \geq (2\ell + 1)^n r_{\min}$.

Proof. The first claim follows at once from the definition. We use the first inequality in (i) inductively to get

$$r_{n+1} \geq 2r_{\text{pack}}(L_{p_n}) + r_n \geq 2r_{\min} + r_n \geq \dots \geq 2nr_{\min}.$$

If T is repulsive, then $r_{\text{pack}}(L_{p_j}) \geq \ell r_j$ for all j , see equation (1), so one gets

$$r_{n+1} \geq 2r_{\text{pack}}(L_{p_n}) + r_n \geq (2\ell + 1)r_n \geq \dots \geq (2\ell + 1)^n r_{\min}.$$

□

The following technical lemma is analogous to Lemma 3.8 in [12]. It states that if a tiling is not repulsive, then one can find arbitrary long sequences of derived privileged patches whose radii grow “slowly”.

Lemma 3.3. If T is not repulsive, then for all $m \in \mathbb{N}$, there exists privileged patches p_0, p_1, \dots, p_m , of radii r_0, r_1, \dots, r_m respectively, such that

(i) $p_j = p_0^{(j)}$, for all $j = 1, \dots, m$,

(ii) $r_m \leq 2r_1$.

Proof. Consider a non repulsive tiling T , and fix an integer $m > 1$. Since the infimum in equation (1) is zero, for any $0 < \epsilon < 1/(8m)$, there is an r_q -patch q of T for which $r_{\text{pack}}(L_q)/r_q < \epsilon$. By FLC there are two copies of q which occur at some markers x and y of tiles in T , satisfying $|x - y| = 2r_{\text{pack}}(L_q)$.

Set $a = r_{\text{pack}}(L_q)$. Consider the largest privileged patch p contained in q , with same marker, and of radius $r \leq r_q$. We must have $r \geq r_q/2$ for otherwise, as p occurs both at x and y , then one of its derived patches p' would have radius $r' \leq r + 2a < r_q/2 + 2\epsilon r_q < r_q$. Hence p' would be contained in q , which contradicts maximality of p in q .

Consider the largest privileged patch $p_0 \subset p$, with same marker, and radius $r_0 < r/2$. Since p_0 occurs both at x and y , it has a derived patch $p_1 \subset p$, with radius $r_1 \leq r_0 + 2a < r/2 + \epsilon r_q < r$, which is thus a proper sub-patch of p . By Lemma 3.1 (iii), we have $p = p_1^{(n-1)} = p_0^{(n)}$ for some $n > 1$. Again p_1 occurs both at x and y , so it has a derived patch $p_2 = p_0^{(2)} \subsetneq p$ of radius $r_2 \leq r_1 + 2a \leq r_0 + 4a$. See Figure 2 for an illustration. We iterate this argument to obtain that there is a patch $p_j = p_0^{(j)}$ with radius $r_j \leq r_0 + 2ja$, for all $j \leq n$. For $j = n$ this last inequality implies $n \geq r/(4a) \geq r_q/(8a) > 1/(8\epsilon) > m$. We have thus build a sequence of privileged patches $p_0, p_1, \dots, p_m, \dots, p_n = p$, whose first $m + 1$ terms give the sequence in (i). But p_0 is the largest privileged patch in p (with same marker and) of radius $r_0 < r/2$, hence $p_1 = p'_0$ has radius $r_1 \geq r/2 = r_n/2 \geq r_m/2$, which proves (ii). □

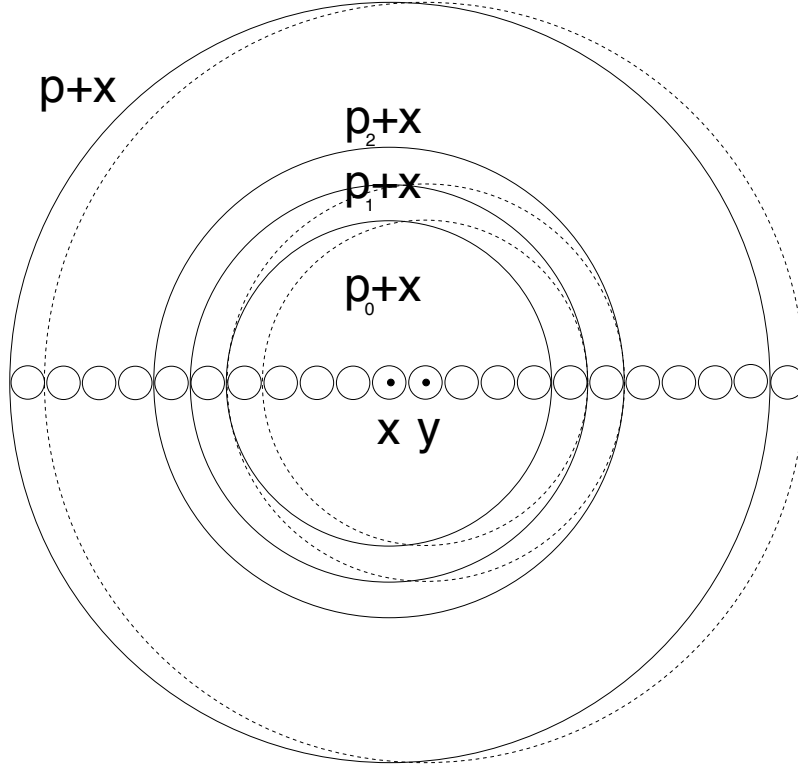


Figure 2: The dotted circles, in order of decreasing radii, are the translates $p + y$, $p_1 + y$, and $p_0 + y$. The small circles along the horizontal axis are occurrences of the same patch, and illustrate the local periodic pattern generated by the overlapping of $p + x$ and $p + y$.

4 The tree of privileged patches and the Connes metrics

We build the tree \mathcal{T} of privileged patches of T inductively as follows:

- (0) the root of \mathcal{T} stands for the empty patch;
- (1) vertices of order 1 stand for privileged patches of order 1 (prototiles), each of which is linked by one edge to the root;
- (n) vertices of order $n > 1$ stand for privileged patches of order n , each of which is linked by one edge to the vertex of order $n - 1$ corresponding to the patch it is derived from.

The tree \mathcal{T} is well-defined by Lemma 3.1 (ii): each vertex of level $n + 1$ is linked to a unique vertex of level n , for all n .

We let $\partial\mathcal{T}$ be the set of infinite rooted path in \mathcal{T} : $\xi = (\xi_n)_{n \geq 0} \in \partial\mathcal{T}$ is a sequence of privileged patches, with ξ_{n+1} derived from ξ_n for all n . Given a vertex $v \in \mathcal{T}$, we let $[v] \subset \partial\mathcal{T}$ be the *cylinder* of v , namely the set of all infinite paths through v .

Proposition 4.1. *The set $\partial\mathcal{T}$ of infinite paths in \mathcal{T} , endowed with the topology of cylinders, is homeomorphic to the discrete hull Ξ .*

Proof. The sets $\partial\mathcal{T}$ and Ξ are easily seen to be isomorphic. Given a tiling T in Ξ , let ξ_0 be the empty patch and ξ_1 the prototile occurring in T at the origin. Since T is repetitive, there is a (unique) privileged patch ξ_2 derived from ξ_1 which occurs in T at the origin. We construct inductively a (unique) sequence of privileged patches occurring at the origin of T , which defines an infinite path in \mathcal{T} . Conversely, by Lemma 3.2 (ii), a sequence of privileged patches in $\partial\mathcal{T}$ defines a unique tiling in Ξ .

A basis for the topology of Ξ is given by the acceptance domains $[p]$ of patches. While cylinders correspond to acceptance domains of privileged patches, hence yield a coarser topology on Ξ . Given a patch p , let p_0 is the greatest privileged patch contained in p , and p_1, \dots, p_k the patches derived from p_0 . Then $[p] \subset [p_1] \cup \dots \cup [p_k]$, so both topologies agree. \square

A *weight function* is any function $\delta : [0, +\infty) \rightarrow (0, 1]$, which decreases and has limit 0 at $+\infty$. A weight function allows us to defined a ultra metric on Ξ , as in equation (2), and to build a spectral triple on $C(\Xi)$ as explained in Section 6. The Connes distance of that spectral triple yields two pseudo-metrics on $\partial\mathcal{T} \simeq \Xi$, which we now define.

Given $\xi, \xi' \in \partial\mathcal{T}$, we let $\xi \tilde{\wedge} \xi'$ denote the vertex at which the paths branch in \mathcal{T} , and $\mathcal{O}(\xi \tilde{\wedge} \xi')$ the order of that vertex. If we identify ξ, ξ' with tilings $T, T' \in \Xi$ by Proposition 4.1, then $\xi \tilde{\wedge} \xi'$ is the greatest common privileged patch which occurs in both T and T' at the origin. The following define two metrics³ on $\partial\mathcal{T}$:

$$d_{\text{inf}}(\xi, \xi') = \begin{cases} \delta(r_{\mathcal{O}(\xi \tilde{\wedge} \xi')}) & \text{if } \xi \neq \xi' \\ 0 & \text{if } \xi = \xi' \end{cases}, \quad (3)$$

and

$$d_{\text{sup}}(\xi, \xi') = d_{\text{inf}}(\xi, \xi') + \sum_{n > \mathcal{O}(\xi \tilde{\wedge} \xi')} \delta(r_n) + \delta(r'_n), \quad (4)$$

where $r_n^{(')}$ is the radius of the patch $\xi_n^{(')}$ (so one has $r_n = r'_n$ for all $n \leq \mathcal{O}(\xi \tilde{\wedge} \xi')$).

Clearly $d_{\text{inf}} \leq d_{\text{sup}}$, and one easily sees that d_{inf} and d_{sup} are Lipschitz equivalent if and only if

$$\exists C > 0, \forall \xi \in \partial\mathcal{T}, \forall m \in \mathbb{N}, \quad \delta(r_m)^{-1} \sum_{k \geq 1} \delta(r_{m+k}) \leq C. \quad (5)$$

5 Characterisation of repulsive tilings

We state our main result. Let T be an aperiodic, repetitive, and FLC tiling of \mathbb{R}^d , as in Definition 2.1. Consider the tree \mathcal{T} of privileged patches of T as in the previous section.

³ d_{inf} is a ultra-metric, d_{sup} is valued in $[0, +\infty]$.

Let $\delta : [0, +\infty) \rightarrow (0, 1]$ be a weight function as in the previous section (decreasing with limit 0 at infinity), and assume that there exists $c_1, c_2 > 0$ such that

$$\delta(ab) \leq c_1 \delta(a) \delta(b), \quad \delta(2a) \geq c_2 \delta(a), \quad \forall a, b \geq 0. \quad (6)$$

Consider the metrics d_{\inf} and d_{\sup} on $\partial\mathcal{T}$ given in equations (3) and (4).

Theorem 5.1. *The following are equivalent:*

- (i) T is repulsive,
- (ii) d_{\inf} and d_{\sup} are Lipschitz equivalent.

Proof. Assume T is repulsive, so $\ell > 0$ in equation (1). Upon rescaling δ , we may assume that $c_1 = 1$ in equation (6), and $\delta(2\ell + 1) < 1$. Pick $m \in \mathbb{N}$ and $\xi \in \partial\mathcal{T}$. Let r_n be the radius of the patch ξ_n , $n \geq 0$. By Lemma 3.2 (ii), for any $k \geq 1$ we have $r_{m+k} \geq (2\ell + 1)^k r_m$. Hence

$$\delta(r_m)^{-1} \sum_{k \geq 1} \delta(r_{m+k}) \leq \delta(r_m)^{-1} \sum_{k \geq 1} \delta((2\ell + 1)^k) \delta(r_m) \leq \sum_{k \geq 1} \delta(2\ell + 1)^k,$$

where the last two inequalities follow from equation (6). The converging geometric series on the right hand side gives a uniform bound in equation (5), which proves that d_{\inf} and d_{\sup} are Lipschitz equivalent.

Assume that T is not repulsive. Fix an integer N (large), and consider a sequence of privileged patches p_0, p_1, \dots, p_m , $m > N$, as in Lemma 3.3. Choose an infinite path $\xi \in \partial\mathcal{T}$ going through the vertices associated with p_0, p_1, \dots, p_m . Upon a change of index, we may assume that ξ_j corresponds to p_j , for $j = 1, \dots, m$. Then

$$\delta(r_1)^{-1} \sum_{k \geq 1} \delta(r_{1+k}) \geq \frac{1}{\delta(r_1)} \sum_{j=2}^m \delta(r_j) \geq \frac{m-1}{\delta(r_1)} \delta(r_m) \geq \frac{N}{\delta(r_1)} \delta(2r_1) \geq Nc_2,$$

where we used that δ decreases, and equation (6). Since N was arbitrary, one cannot bound the series on the left hand side. So there exists no uniform bound in equation (5), and thus d_{\inf} and d_{\sup} are not Lipschitz equivalent. \square

6 The spectral triple

For the sake of completeness, we remind the reader of the spectral triple on $C(\Xi) \cong C(\partial\mathcal{T})$ whose Connes distance yields d_{\inf} and d_{\sup} . The construction in [11] is given for any tree, and in [12] for the tree of privileged words of a $1d$ -subshift, which we rewrite here for the tree of privileged patches defined in Section 4. These constructions are related to other spectral triples build for metric spaces [19, 20, 2] or more specifically fractals [7, 8, 3] and ultrametric Cantor sets [16]. We refer the reader to [11] and [12] for details and proofs.

We consider the tree $\mathcal{T} = (\mathcal{T}^0, \mathcal{T}^1)$ of privileged patches, and a weight δ as in Section 4. We add *horizontal edges* \mathcal{H} to the graph \mathcal{T} : $\mathcal{H} = \cup_{n \geq 1} \mathcal{H}_n$, and \mathcal{H}_n is a set of oriented edges between vertices of level n in \mathcal{T} defined as follows. If $u_1, u_2 \in \mathcal{T}$, then there is one horizontal edge $h \in \mathcal{H}_n$ with source $s(h) = u_1$ and range $r(h) = u_2$, if and only if u_1 and u_2 stand for two distinct privileged patches of order n both of which are derived from the same privileged patch of order $n - 1$. Given any such h , there is then an edge $h^{\text{op}} \in \mathcal{H}$ with reverse orientation: $r(h^{\text{op}}) = s(h)$ and $s(h^{\text{op}}) = r(h)$, and $(h^{\text{op}})^{\text{op}} = h$. We fix an orientation of \mathcal{H} , and write the decomposition into positively and negatively oriented edges $\mathcal{H} = \mathcal{H}_+ \cup \mathcal{H}_-$.

A *choice* is a map $\tau : \mathcal{T}^0 \rightarrow \partial\mathcal{T}$ such that $\tau(v)$ is an infinite path through vertex v . The *approximation graph* $G_\tau = (V, E)$ is defined by

$$V = \tau(\mathcal{T}^0), \quad E = \tau \times \tau(\mathcal{H}).$$

The orientation on \mathcal{H} is carried over to $E = E_+ \cup E_-$. We endow G_τ with the graph metric induced by the weight δ : for $e = \tau \times \tau(h) \in E$ we set the length of e to be $\ell(e) = \delta(r_h)$, where r_h is the radius of the privileged patch from which $s(h)$ and $r(h)$ are derived. The set of vertices V is dense in $\partial\mathcal{T}$, and the set of edges E encodes adjacencies according to the choice τ .

We consider the spectral triple associated with the approximation graph $G_\tau = (V, E)$: $(C(\partial\mathcal{T}), \ell^2(E), \pi_\tau, D)$. The C^* -algebra $C(\partial\mathcal{T})$ of continuous functions over $\partial\mathcal{T}$ is faithfully represented on the Hilbert space $\ell^2(E)$ by

$$\pi_\tau(f)\varphi(e) = f(s(e))\varphi(e).$$

The Dirac D is the unbounded operator on $\ell^2(E)$, with compact resolvent, given by

$$D\varphi(e) = \frac{1}{\ell(e)}\varphi(e^{\text{op}}).$$

The “non commutative derivation” is the finite difference operator

$$[D, \pi_\tau(f)]\varphi(e) = \frac{f(s(e)) - f(r(e))}{\ell(e)}\varphi(e^{\text{op}}),$$

which is bounded over the pre- C^* -algebra $C_{\text{Lip}}(\partial\mathcal{T})$ of Lipschitz continuous functions over $\partial\mathcal{T}$. The Connes distance of the spectral triple is a pseudo-metric over $\partial\mathcal{T}$ which reads

$$\begin{aligned} d_\tau(\xi, \xi') &= \sup_{f \in C(\partial\mathcal{T})} \left\{ |f(\xi) - f(\xi')| : \|[D, \pi_\tau(f)]\| \leq 1 \right\} \\ &= \sup_{f \in C(\partial\mathcal{T})} \left\{ |f(\xi) - f(\xi')| : |f(s(e)) - f(r(e))| \leq \ell(e), \forall e \in E \right\} \end{aligned}$$

where the norm is the operator norm on $\ell^2(E)$. It is an extension of the graph metric of G_τ to $\partial\mathcal{T}$. If it is continuous for the topology of \mathcal{T} , it reads explicitly:

$$d_\tau(\xi, \xi') = d_{\text{inf}}(\xi, \xi') + \sum_{n > \mathcal{O}(\xi \tilde{\wedge} \xi')} b_\tau(\xi_n)\delta(r_n) + b_\tau(\xi'_n)\delta(r'_n),$$

where $\xi^{(\prime)} = (\xi_n^{(\prime)})$, $r_n^{(\prime)}$ is the radius of the patch $\xi_n^{(\prime)}$, and for $\eta \in \partial\mathcal{T}$, $b_\tau(\eta_n) = 1$ if η and $\tau(\eta_n)$ branch at η_n , and $b_\tau(\eta_n) = 0$ otherwise. The infimum and supremum of d_τ over choices τ yield the metrics d_{\inf} of equation (3) and d_{\sup} and (4) respectively.

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